# Analytical Inversion of a Class of Infinitely Dimensioned Matrices Encountered in Some Microwave Problems

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*Abstract*— **A method is presented for analytically inverting a class of infinitely dimensioned matrices that are usually encountered in the analysis of microwave problems by applying the generalized spectral domain (GSD) method. It is shown that numerically inverting the truncated matrix and truncating the analytically inverted matrix leads to different results, which converge to each other very slowly. CPU time and storage requirements of a number of algorithms based on the GSD method can consequently be improved by making use of the analytical inversion of such matrices.**

## I. Introduction

The GSD method as presented in [1], [2] is mainly applied to shielded structures. The structure is first divided into one or more cavity resonators which are coupled together and to one or more waveguides via coupling apertures. The coupling apertures are short circuited and the nonvanishing tangential electric field there is restored by introducing magnetic surface currents. The now completely shielded cavity resonators and short circuited waveguides are then analyzed by expanding the electromagnetic wave excited by the magnetic surface currents in terms of the empty cavity modes and the empty waveguide eigenmodes, respectively. If a cavity resonator (a waveguide) contains a dielectric or a magnetic inhomogeneity (e.g., a dielectric insert), the cavity modes (the waveguide eigenmodes) become coupled. This coupling is characterized by a coupling matrix with triply (doubly) infinite dimension. In order to find an expression for the overall scattering matrix (of the whole structure) the field expansion coefficients in the different regions must be eliminated which necessitates the inversion of the coupling matrix.

It is the aim of this letter to present a method for the analytical inversion of coupling matrices.

## II. Basic Formulation

Let us have a volume $V$ enclosed by a perfectly conducting surface $S$. Complete function-sets, in terms of which an arbitrary function defined on $V$ can be expanded, are then discrete. Let the orthonormal set $\{u_n\}$ be one of these sets.

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One has then

$$\int_V u_n(\boldsymbol{r}) u_m^*(\boldsymbol{r})\, dV = \delta_{nm} \tag{1}$$

$$\sum_n^{\infty} u_n(\boldsymbol{r}) u_m^*(\boldsymbol{r}') = \delta(\boldsymbol{r} - \boldsymbol{r}'), \tag{2}$$

where $\delta_{nm}$ is the Kronecker delta and $\delta(\boldsymbol{r} - \boldsymbol{r}')$ is the Dirac delta function.

Let us now consider the two matrices $[A]$ and $[B]$ with elements

$$A_{nm} = \int_V f(\boldsymbol{r}) u_n(\boldsymbol{r}) u_m^*(\boldsymbol{r})\, dV, \tag{3}$$

$$B_{nm} = \int_V f^{-1}(\boldsymbol{r}) u_n(\boldsymbol{r}) u_m^*(\boldsymbol{r})\, dV. \tag{4}$$

The product matrix $[P] = [A][B]$ has the following elements:

$$\begin{aligned}
P_{nm} &= \sum_k^{\infty} A_{nk} B_{km} \\
&= \sum_k^{\infty} \left[\int_V f(\boldsymbol{r}) u_n(\boldsymbol{r}) u_k^*(\boldsymbol{r})\, dV\right] \cdot \left[\int_V f^{-1}(\boldsymbol{r}') u_k(\boldsymbol{r}') u_m^*(\boldsymbol{r}')\, dV'\right] \\
&= \int_V \int_V f(\boldsymbol{r}) f^{-1}(\boldsymbol{r}') u_n(\boldsymbol{r}) u_m^*(\boldsymbol{r}') \left[\sum_k^{\infty} u_k(\boldsymbol{r}') u_k^*(\boldsymbol{r})\right] dV'dV \\
&= \int_V \int_V f(\boldsymbol{r}) f^{-1}(\boldsymbol{r}') u_n(\boldsymbol{r}) u_m^*(\boldsymbol{r}') \delta(\boldsymbol{r}' - \boldsymbol{r})\, dV'dV \quad \text{(from (2))} \\
&= \int_V f(\boldsymbol{r}) f^{-1}(\boldsymbol{r}) u_n(\boldsymbol{r}) u_m^*(\boldsymbol{r})\, dV \\
&= \delta_{nm} \quad \text{(from (1))}.
\end{aligned}$$

This leads then to

$$[B] = [A]^{-1}. \tag{5}$$

In the following, we will restrict the analysis to the coupling by a dielectric inhomogeneity.

## III. Three-Dimensional (Resonator) Problems

A dielectric inhomogeneity (represented by the spatially dependent relative dielectric constant $\epsilon_r(\boldsymbol{r})$) couples the electric fields of the different empty cavity modes. These modes are the divergence-free resonance modes $\{\boldsymbol{E}_n\}$ and the curl-free modes $\{\boldsymbol{F}_n\}$ (see [3]). These modes are complete and orthonormalized according to

$$\int_V \boldsymbol{E}_n \cdot \boldsymbol{E}_m^* \, dV = \delta_{nm}, \tag{6}$$

$$\int_V \boldsymbol{F}_n \cdot \boldsymbol{F}_m^* \, dV = \delta_{nm}, \tag{7}$$

$$\int_V \boldsymbol{E}_n \cdot \boldsymbol{F}_m^* \, dV = 0, \tag{8}$$

where $V$ is the volume of the cavity.

Referring to [2], coupling between the resonance modes is represented by a matrix $\left[C^{EE}\right]$ with elements

$$C_{nm}^{EE} = \int_V \epsilon_r^{-1}(\boldsymbol{r}) \boldsymbol{E}_n \cdot \boldsymbol{E}_m^* \, dV. \tag{9}$$

The whole coupling is, however, represented by a matrix $[C]$, which is given by

$$[C] = \begin{bmatrix} \left[C^{EE}\right] & \left[C^{EF}\right] \\ \left[C^{FE}\right] & \left[C^{FF}\right] \end{bmatrix}, \tag{10}$$

where

$$C_{nm}^{EF} = \int_V \epsilon_r^{-1}(\boldsymbol{r}) \boldsymbol{E}_n \cdot \boldsymbol{F}_m^* \, dV, \tag{11}$$

$$C_{nm}^{FE} = \int_V \epsilon_r^{-1}(\boldsymbol{r}) \boldsymbol{F}_n \cdot \boldsymbol{E}_m^* \, dV, \tag{12}$$

$$C_{nm}^{FF} = \int_V \epsilon_r^{-1}(\boldsymbol{r}) \boldsymbol{F}_n \cdot \boldsymbol{F}_m^* \, dV. \tag{13}$$

The analytical inversion $[C]^{-1}$ is obtained by simply replacing $\epsilon_r^{-1}$ in (9)–(13) by $\epsilon_r$. Truncating the analytically inverted matrix (for the sake of numerical implementation) is much more accurate than numerically inverting the truncated matrix.

## IV. Two-Dimensional (Waveguide) Problems

Inhomogeneously filled waveguides have been analyzed in [1] by applying the generalized spectral domain method. The divergence-free TE set $\left\{\hat{\boldsymbol{k}} \times \boldsymbol{\nabla}_t h_{zn}\right\}$ and the curl-free TM set $\{\boldsymbol{\nabla}_t e_{zn}\}$ are (together) complete. They are orthonormalized according to

$$\int_S \boldsymbol{\nabla}_t e_{zn} \cdot \boldsymbol{\nabla}_t e_{zm} \, dS = \delta_{nm}, \tag{14}$$

$$\int_S \boldsymbol{\nabla}_t h_{zn} \cdot \boldsymbol{\nabla}_t h_{zm} \, dS = \delta_{nm}, \tag{15}$$

$$\int_S (\boldsymbol{\nabla}_t e_{zn} \times \boldsymbol{\nabla}_t h_{zm}) \cdot \hat{\boldsymbol{k}} \, dS = 0, \tag{16}$$

where $S$ is the cross section of the waveguide and $\hat{\boldsymbol{k}}$ is the unit vector in axial direction.

Coupling between the different eigenmodes by a dielectric inhomogeneity is represented by the coupling matrix $[C]$ which is given by

$$[C] = \begin{bmatrix} \left[C^{ee}\right] & \left[C^{eh}\right] \\ \left[C^{he}\right] & \left[C^{hh}\right] \end{bmatrix}, \tag{17}$$

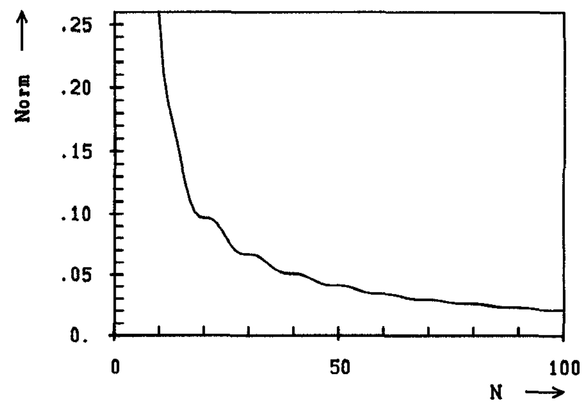


Fig. 1. Norm $\|\left[([A]_N)^{-1}\right]_M - [B]_M\|$ versus $N$. Parameter: $a = \pi$, $a_0 = (\pi/10)$, $c = 10$, $M = 10$.

where

$$C_{nm}^{ee} = \int_S \epsilon_r \boldsymbol{\nabla}_t e_{zn} \cdot \boldsymbol{\nabla}_t e_{zm} \, dS, \tag{18}$$

$$C_{nm}^{eh} = \int_S \epsilon_r (\boldsymbol{\nabla}_t e_{zn} \times \boldsymbol{\nabla}_t h_{zm}) \cdot \hat{\boldsymbol{k}} \, dS, \tag{19}$$

$$C_{nm}^{he} = \int_S \epsilon_r (\boldsymbol{\nabla}_t e_{zm} \times \boldsymbol{\nabla}_t h_{zn}) \cdot \hat{\boldsymbol{k}} \, dS, \tag{20}$$

$$C_{nm}^{hh} = \int_S \epsilon_r \boldsymbol{\nabla}_t h_{zn} \cdot \boldsymbol{\nabla}_t h_{zm} \, dS, \tag{21}$$

where $\epsilon_r$ depends on the transverse coordinates (but is axially invariant). Again the analytic inverse of $[C]$ is obtained by replacing $\epsilon_r$ in (18)–(21) by $\epsilon_r^{-1}$.

## V. One-Dimensional Problem and Numerical Results

For the one-dimensional segment $[0, a]$ both sets

$$\left\{\sqrt{\frac{2}{a}} \sin\!\left(\frac{n\pi x}{a}\right)\right\} \quad \text{and} \quad \left\{\sqrt{\frac{2}{a(1+\delta_{n0})}} \cos\!\left(\frac{n\pi x}{a}\right)\right\}$$

are complete and orthonormal. The first set will be taken to demonstrate the present method. The two matrices $[A]$, and $[B]$ are given by

$$A_{nm} = \frac{2}{a} \int_0^a f(x) \sin\!\left(\frac{n\pi x}{a}\right) \sin\!\left(\frac{m\pi x}{a}\right) dx$$

$$B_{nm} = \frac{2}{a} \int_0^a f^{-1}(x) \sin\!\left(\frac{n\pi x}{a}\right) \sin\!\left(\frac{m\pi x}{a}\right) dx.$$

The numerical inversion of the coupling matrix $[A]$ has been carried out with the upper $(N \times N)$ corner of $[A]$ (denoted by $[A]_N$). The upper $(M \times M)$ $(M \leq N)$ corner of $([A]_N)^{-1}$ (denoted by $\left[([A]_N)^{-1}\right]_M$) has been compared to the upper $(M \times M)$ corner of the analytic inverse $[B]$ (denoted by $[B]_M$) of $[A]$ for a step function

$$f(x) = \begin{cases} c & \text{on} \quad 0 \leq x \leq a_0 \\ 1 & \text{on} \quad a_0 < x \leq a \end{cases}.$$

Fig. 1 shows the norm $\|\left[([A]_N)^{-1}\right]_M - [B]_M\|$ versus $N$.

It is easily seen that the convergence is extremely poor with respect to $N$. This emphasizes using the truncation of the analytic inverse (which is more accurate) and not the numerical inverse of the truncated matrix in the numerical algorithms.

## VI. Conclusion

The analytical inversion of a class of triply-, doubly- and simply-infinite dimensional matrices, which are usually encountered in the GSD analysis of shielded structures is presented. A better convergence, which leads to a drastic reduction in CPU time and storage requirements for a number of GSD algorithms is achieved if the numerical inversion of the truncated matrix is replaced by the truncation of the analytically inverted matrix.

## References


[1] A. S. Omar and K. Schünemann, "Complex and backward-wave modes in inhomogeneously filled waveguides," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-35, pp. 268–275, 1987.

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[3] R. E. Collin, *Foundations for Microwave Engineering.* New York: McGraw-Hill, 1966.